UNIVERSIDADE FEDERAL DE SANTA CATARINA
VII Encontro de Professores da ORM
PET – MATEMÁTICA

Soluções: Desigualdade

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1. $(a-b)^2 \geq 0 \Rightarrow$

   $a^2 + b^2 \geq 2ab \Rightarrow$

   $2a^2 + 2b^2 \geq a^2 + 2ab + b^2 = (a+b)^2 \Rightarrow$

   $2(a^2 + b^2) \geq (a+b)^2 \Rightarrow$

   $\dfrac{a^2+b^2}{a+b} \geq \dfrac{a+b}{2}$

2. Sejam $a, b, c \geq 0$

   (a) Temos que:

   $\dfrac{a+b}{2} \geq \sqrt{ab} \quad (I)$

   $\dfrac{a+c}{2} \geq \sqrt{ac} \quad (II)$

   $\dfrac{b+c}{2} \geq \sqrt{bc} \quad (III)$

   Multiplicando $(I), (II), (III)$, temos:

   $\left(\dfrac{a+b}{2}\right)\left(\dfrac{a+c}{2}\right)\left(\dfrac{b+c}{2}\right) \geq \sqrt[2]{a^2 \cdot b^2 \cdot c^2} = abc$

   Segue-se ainda que:

   $(a+b)(a+c)(b+c) \geq 8abc$

   (b) $(ab-bc)^2 \geq 0 \quad (I)$

   $(ab-ac)^2 \geq 0 \quad (II)$

   $(ac-bc)^2 \geq 0 \quad (III)$

   De (I), segue-se:

   $a^2b^2 + b^2c^2 \geq 2ab^2c \quad (IV)$

   De (II), temos:

   $a^2b^2 + a^2c^2 \geq 2a^2bc \quad (V)$

   De (III), segue-se:

$a^2c^2 + b^2c^2 \geq 2abc^2 \quad (VI)$

Somando $(IV), (V), (VI)$, temos:

$2a^2b^2 + 2a^2c^2 + 2b^2c^2 \geq 2ab^2c + 2a^2bc + 2abc^2 \Rightarrow$

$2(a^2b^2 + a^2c^2 + b^2c^2) \geq 2abc(a + b + c) \Rightarrow$

$a^2b^2 + a^2c^2 + b^2c^2 \geq abc(a + b + c)$

(c) Partimos da questão já demonstrada que:

$a^2 + b^2 + c^2 \geq ab + ac + bc$

Segue-se:

$a(a^2 + b^2 + c^2) \geq a(ab + ac + bc) \quad (I)$

$b(a^2 + b^2 + c^2) \geq b(ab + ac + bc) \quad (II)$

$c(a^2 + b^2 + c^2) \geq c(ab + ac + bc) \quad (III)$

Desenvolvendo as inequações e somando (I), (II), (III), temos:

$a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3 \geq a^2b+a^2c+abc+ab^2+abc+b^2c+cab+ac^2+bc^2$

Usando a lei do cancelamento da soma, segue-se:

$a^3 + b^3 + c^3 \geq 3abc$

(d) $\dfrac{a+b+c}{abc} = \dfrac{abc(a+b+c)}{a^2b^2c^2} \leq \dfrac{a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2c^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

3. Sejam $a, b, c > 0$

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = \frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3 =$$
$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3$$
$$\frac{1}{2}[(a+b) + (a+c) + (b+c)]\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3 =$$

$$= \frac{1}{2}\left(\frac{a+b}{b+c} + \frac{a+b}{a+c} + \frac{a+c}{b+c} + \frac{a+c}{a+b} + \frac{b+c}{a+c} + \frac{b+c}{a+b} + 3\right) - 3 =$$

$$= \frac{1}{2}\left(\left(\frac{a+b}{b+c} + \frac{b+c}{a+b}\right) + \left(\frac{a+b}{a+c} + \frac{a+c}{a+b}\right) + \left(\frac{a+c}{b+c} + \frac{b+c}{a+c}\right) + 3\right) - 3 \geq *$$

Relembrando que para todo $x > 0$, então $x + \dfrac{1}{x} \geq 2$

$$* \geq \frac{1}{2}(2 + 2 + 2 + 2 + 3) - 3 = \frac{9}{2} - 3 = \frac{3}{2}$$

Portanto:

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}$$

4. Sejam ${x_1}^2+{x_2}^2+{x_3}^2\leq 1 \quad \text{e} \quad a_{ij}\leq M$

   Temos que:

   $$a_{11}{x_1}^2+a_{22}x_2^2+a_{33}x_3^2+a_{12}x_1x_2+a_{13}x_1x_3+a_{23}x_2x_3+a_{21}x_2x_1+a_{31}x_3x_1+a_{32}x_3x_2\leq$$

   $$M(x_1^2+x_2^2+x_3^2)+M(x_1x_2+x_1x_3+x_2x_3)+M(x_2x_1+x_3x_1+x_3x_2)\leq M(x_1^2+x_2^2+x_3^2)+$$

   $$M(x_1^2+x_2^2+x_3^2)+M(x_1^2+x_2^2+x_3^2)\leq 3M.$$

   Relembrando novamente que:

   $$ab+ac+bc\leq a^2+b^2+c^2$$

5. Como $a^2+b^2+c^2=1$, então:

   $$ab+ac+bc\leq a^2+b^2+c^2=1$$

   Por outro lado:

   $$(a+b+c)^2\geq 0\Rightarrow$$

   $$a^2+b^2+c^2+2ab+2ac+2bc\geq 0\Rightarrow$$

   $$2(ab+ac+bc)\geq -1\Rightarrow$$

   $$ab+ac+bc\geq -\frac{1}{2}$$

6.

   $$\frac{1+a_1}{2}\geq\sqrt{1\cdot a_1}=\sqrt{a_1}$$
   $$\frac{1+a_2}{2}\geq\sqrt{a_2}$$
   $$\vdots$$
   $$\frac{1+a_n}{2}\geq\sqrt{a_n}$$

   $$\Rightarrow\left(\frac{1+a_1}{2}\right)\cdot\left(\frac{1+a_2}{2}\right)\ldots\left(\frac{1+a_n}{2}\right)\geq\sqrt{a_1-a_n}=1\Rightarrow$$
   $$(1+a_1)-(1+a_n)\geq 2^n$$

7.

Sejam a, b, c lados de um triângulo

$ab + ac + bc \leq a^2 + b^2 + c^2 \ \ \forall\, a, b, c \geq 0$

Por outro lado:

$a \geq \mid b - c \mid \Rightarrow \ a^2 \geq (b - c)^2 \qquad (I)$

$b \geq \mid a - c \mid \Rightarrow \ b^2 \geq (a - c)^2 \qquad (II)$

$c \geq \mid a - b \mid \Rightarrow \ c^2 \geq (a - b)^2 \qquad (III)$

Somando (I),(II),(III), temos:

$a^2 + b^2 + c^2 \geq (b - c)^2 + (a - c)^2 + (a - b)^2 \ =$

$= 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc$

$0 \geq a^2 + b^2 + c^2 - 2(ab + ac + bc) \ \Rightarrow$

$2(ab + ac + bc) \geq a^2 + b^2 + c^2$